The Seventh Small Triangle Meeting — Snina, September 2–4, 2005

Combined effects of compressibility and helicity on the scaling regimes of a passive scalar advected by turbulent velocity field with finite correlation time

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Abstract

The influence of compressibility and helicity on the stability of the scaling regimes of a passive scalar advected by a Gaussian velocity field with finite correlation time is investigated by the field theoretic renormalization group within two-loop approximation. The influence of helicity and compressibility on the scaling regimes is discussed as a function of the exponents ε and η, where ε characterizes the energy spectrum of the velocity field in the inertial range $E \propto k^{1-2\varepsilon}$, and η is related to the correlation time at the wave number k which is scaled as $k^{-2+\eta}$. The restrictions given by nonzero compressibility and helicity on the regions with stable infrared fixed points which correspond to the stable infrared scaling regimes are discussed. A special attention is paid to the case of so-called frozen velocity field when the velocity correlator is time independent. In this case, explicit inequalities which must be fulfilled in the plane $\varepsilon - \eta$ are determined within two-loop approximation.

Introduction

One of the main problems in the modern theory of fully developed turbulence is to verify the validity of the basic principles of Kolmogorov-Obukhov (KO) phenomenological theory and their consequences within the framework of a microscopic model [1, 2]. On the other hand, recent experimental, numerical and theoretical studies signify the existence of deviations from the well-known Kolmogorov scaling behavior. The scaling behavior of the velocity fluctuations with exponents, which values are different from Kolmogorov ones, is known as anomalous and is associated with intermittency phenomenon [2]. Even thought the understanding of the intermittency and anomalous scaling within the theoretical description of the fluid turbulence on basis of the "first principles", i.e., on the stochastic Navier-Stokes equation, still remains an open problem, considerable progress has been achieved in the studies of the simplified model systems which share some important properties of the real turbulence.

The crucial role in these studies is played by models of advected passive scalar field [3]. Maybe the most known model of this type is a simple model of a passive scalar quantity advected by a random Gaussian velocity field, white in time and self-similar in space, the so-called Kraichnan's rapid-change model [4]. It was shown by both natural and numerical experimental investigations that the deviations from the predictions of the classical KO phenomenological theory is even more strongly displayed for a passively advected scalar field than for the velocity field itself (see, e.g., [5] and references cited therein). At the same time, the problem of passive advection is much more easier to be consider from theoretical point of view. There, for the first time, the anomalous scaling was established on the basis of a microscopic model [6], and corresponding anomalous exponents was calculated within controlled approximations (see review [5] and references therein).

In paper [7] the field theoretic renormalization group (RG) and operator-product expansion (OPE) were used in the systematic investigation of the rapid-change model. It was shown that within the field theoretic approach the anomalous scaling is related to the very existence of so-called "dangerous" composite operators with negative critical dimensions in OPE (see, e.g., [8, 9] for details).

Afterwards, various generalized descendants of the Kraichnan model, namely, models with inclusion of large and small scale anisotropy [10], compressibility [11] and finite correlation time of the velocity field [12, 13] were studied by the field theoretic approach. General conclusion is: the anomalous scaling, which is the most important feature of the Kraichnan rapid change model, remains valid for all generalized models.

In paper [12] the problem of a passive scalar advected by the Gaussian self-similar velocity field with finite correlation time [14] was studied by the field theoretic RG method. There, the systematic study of the possible scaling regimes and anomalous behavior was present at one-loop level. The two-loop corrections to the anomalous exponents were obtained in [15]. In paper [13] the influence of compressibility on the problem studied in [12] was analyzed. The effects of the presence of helicity in the system on the scaling regimes and anomalous dimensions were studied in [16, 17] within of two-loop approximation. It was shown that although the separate composite operators which define anomalous dimensions strongly depend on the helicity parameter the resulting two-loop contributions to the critical dimensions of the structure functions are independent of helicity. This rather intriguing result can have at least two independent explanations. First, it can be simply a two-loop result which will be changed when one will study three-loop approximation. Second, more interesting conclusion (but it is only a speculation for now) is that this situation will be held at each level of the perturbation theory while the assumption of incompressibility or isotropy will be supposed. From this point of view, the investigation of the compressible system together with assumption of helicity of the system within two-loop approximation can give an interesting answer. For this purpose in [18] the influence of compressibility of the model on the stability of the scaling regimes was studied and the restrictions on the parameter space were analyzed. In what follows, we shall continue these studies and our aim will be to find combine effects of helicity and compressibility on the stability of the scaling regimes in two-loop approximation. It can be consider as the starting point for the subsequent investigation of the anomalous scaling of the correlator or structure functions of a passive scalar.

Formulation of the model

The advection of a passive scalar field is described by equation

$$\partial_t \theta + (\mathbf{v}\cdot\partial)\theta = \nu_0 \nabla \theta + f^\theta \,, \tag{1}$$

where $\theta(x) = \theta(t, \mathbf{x})$ is a passive scalar field, ν_0 is the molecular diffusivity coefficient, $\mathbf{v}(x)$ is compressible velocity field, and $f^\theta \equiv f^\theta(x)$ is a Gaussian random noise with zero mean and correlation function

$$\langle f^\theta(x) f^\theta(x')\rangle = \delta(t-t') C(\mathbf{r}/\tilde{L}), \;\; \mathbf{r} = \mathbf{x} - \mathbf{x}', \tag{2}$$

where parentheses $\langle ...\rangle$ denote average over corresponding statistical ensemble. The noise maintains the steady-state of the system but its concrete form is not essential. $\tilde{L}$ denotes an integral scale related to the stirring.

In real problems the velocity field $\mathbf{v}(x)$ satisfies stochastic Navier-Stokes equation but we shall suppose that the velocity field obeys a Gaussian distribution with zero mean and correlator

$$\langle v_i(x) v_j(x')\rangle = \int \frac{d\omega}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^d} R_{ij}(\mathbf{k}) D_v(\omega, k) e^{-i\omega(t-t') + i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \,. \tag{3}$$

Here $R_{ij}(\mathbf{k}) = P_{ij}(\mathbf{k}) + Q_{ij}(\mathbf{k}) + H_{ij}(\mathbf{k})$, where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ is the non-helical transverse projector, $Q_{ij}(\mathbf{k}) = \alpha k_i k_j / k^2$ is longitudinal projector with the compressibility parameter $\alpha \geq 0$,

and $H_{ij}(\mathbf{k}) = i\rho\varepsilon_{ijl}k_l/k$ is the helical transverse projector with helicity parameter $\rho \in \langle 0, 1\rangle$ (ε_{ijl} is Levi-Civita's completely antisymmetric tensor of rank 3, and $k \equiv |\mathbf{k}|$). d is the dimensionality of the $\mathbf{x}$ space which must be taken $d = 3$ in the helical case ($\rho > 0$). The function D_v is taken in the form

$$D_v(\omega, k) = \frac{g_0\nu_0^3 k^{4-d-2\varepsilon-\eta}}{\omega^2 + (u_0\nu_0 k^{2-\eta})^2}\,, \tag{4}$$

where $g_0 > 0$ and the exponent ε describe the equal-time velocity correlator (the energy spectrum). On the other hand, the constant u_0 and the exponent η are related to the frequency ω, characteristic of the mode k. In our model the exponents ε and η play the role of the RG expansion parameters. The model contains two special cases. In the limit $u_0 \to \infty$ and $g_0/u_0^2 = const.$ one obtains the so-called rapid-change model, and the limit $u_0 \to 0$ and $g_0/u_0 = const.$ corresponds to the case of a "frozen" velocity field [12, 13].

The stochastic problem (1)-(3) is equivalent to the field theoretic model of the set of fields $\Phi \equiv \{\theta, \theta', \mathbf{v}\}$ (see, e.g., [8]) with action functional

$$\begin{aligned} S(\Phi) = & -\frac{1}{2}\int dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2}\; v_i(t_1,\mathbf{x_1})[D^v_{ij}(t_1,\mathbf{x_1};t_2,\mathbf{x_2})]^{-1}v_j(t_2,\mathbf{x_2}) \\ & + \int dt\, d^d\mathbf{x}\; \theta'\,[-\partial_t\theta - v_i\partial_i\theta + \nu_0\triangle\theta]\,, \end{aligned} \tag{5}$$

where the unimportant term related to the noise (2) is omitted, θ' is an auxiliary scalar field, and summations are implied over the vector indices. The second line in (5) represents the Martin-Siggia-Rose action for the stochastic problem (1) at fixed velocity field $\mathbf{v}$, and the first line describes the Gaussian averaging over $\mathbf{v}$ defined by the correlator D^v in (3) and (4).

The functional formulation (5) means that statistical averages of random quantities in the stochastic problem defined by (1) and (3) corresponds to functional averages with the weight $S(\Phi)$.

UV renormalization, fixed points and scaling regimes

Using the standard analysis of quantum field theory one can find that the UV divergences are only present in the one-particle-irreducible Green functions $\langle\theta'\theta\rangle_{1-ir}$ and $\langle\theta'\theta\mathbf{v}\rangle_{1-ir}$. The renormalized action functional has the following form

$$\begin{aligned} S(\Phi) = & -\frac{1}{2}\int dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2} v_i(t_1,\mathbf{x_1})[D^v_{ij}(t_1,\mathbf{x_1};t_2,\mathbf{x_2})]^{-1}v_j(t_2,\mathbf{x_2}) \\ & + \int dt\, d^d\mathbf{x}\; \theta'\,[-\partial_t\theta - Z_2 v_i\partial_i\theta + \nu Z_1\triangle\theta]\,, \end{aligned} \tag{6}$$

The dimensionless parameters g, u and ν are the renormalized counterparts of the bare parameters (denoted by 0). Z_1 and Z_2 are the renormalization constants of our model. We shall calculate them in 2-loop approximation in the minimal subtraction scheme. Our model is multiplicatively renormalizable which is represented by the following relations between renormalized and bare parameters (see, e.g., [12, 13]):

$$\nu_0 = \nu Z_\nu,\;\; g_0 = g\mu^{2\varepsilon+\eta}Z_g,\;\; u_0 = u\mu^\eta Z_u,\;\; \mathbf{v} \to Z_v\mathbf{v}. \tag{7}$$

where

$$Z_\nu = Z_1,\;\; Z_u = Z_1^{-1},\;\; Z_g = Z_2^2 Z_1^{-3},\;\; Z_v = Z_2. \tag{8}$$

and μ is the renormalization mass. One and two-loop Feynman diagrams which contribute to the renormalization constants Z_1 and Z_2 can be found in [18]. They have the following form

$$Z_1 = \frac{g}{\varepsilon}A_1 + \frac{g^2}{\varepsilon}\left(\frac{1}{\varepsilon}B_1 + C_1\right),\quad Z_2 = \frac{g}{\varepsilon}A_2 + \frac{g^2}{\varepsilon}\left(\frac{1}{\varepsilon}B_2 + C_2\right), \tag{9}$$

where the one-loop contributions are (in the MS-scheme)

$$A_1 = -\frac{S_d}{(2\pi)^d}\frac{1}{4u(1+u)^2}\frac{(1+u)(d-1+\alpha)-2\alpha}{d}, \quad A_2 = \frac{S_d}{(2\pi)^d}\frac{\alpha}{4u(1+u)^2}, \tag{10}$$

where S_d is d dimensional sphere given by the expression $S_d = 2\pi^{d/2}/\Gamma(d/2)$. Two-loop contributions will not be shown explicitly here (the explicit form of the corresponding expressions in the non-helical case can be found in [18], and the explicit form of the helical two-loop contribution to the Z_1 can be found in [16]).

From the renormalization constants (9) one can define the corresponding anomalous dimensions $\gamma_i = \mu\partial_\mu \ln Z_i$ and β functions for all invariant charges ($X = g, u$): $\beta_X = \mu\partial_\mu X$. In our case we have

$$\gamma_1 \equiv \mu\partial_\mu \ln Z_1 = -2(gA_1 + 2g^2C_1), \quad \gamma_2 \equiv \mu\partial_\mu \ln Z_2 = -2(gA_2 + 2g^2C_2), \tag{11}$$

and

$$\beta_g \equiv \mu\partial_\mu g = g(-2\varepsilon - \eta + 3\gamma_1 - 2\gamma_2), \quad \beta_u \equiv \mu\partial_\mu u = u(-\eta + \gamma_1). \tag{12}$$

Now one has all necessary tools to begin with analysis of possible scaling regimes of the model. It is well known that possible scaling regimes of a renormalizable model are associated with the infrared (IR) stable fixed points of the corresponding RG equations. The fixed points are determined from the requirement that all β functions are vanish: $\beta_X(X_*) = 0$, where $X = g, u$ and X_* are coordinates of the corresponding fixed point. The type of the fixed point is given by the matrix of the first derivatives $\Omega_{ij} = \partial\beta_i/\partial X_j$. The fixed point is IR stable if all the eigenvalues of the matrix Ω are positive (precisely their real parts). In our model we have five types of possible scaling regimes related to fixed points of the model.

First of all, we shall investigate the rapid-change limit: $u \to \infty$. In this regime, it is necessary to make transformation to new variables, namely, $w \equiv 1/u$, and $g' \equiv g/u^2$, with the corresponding changes in the β functions:

$$\beta_{g'} = g'(\eta - 2\varepsilon + \gamma_1 - 2\gamma_2), \quad \beta_w = w(\eta - \gamma_1). \tag{13}$$

In this case we have two fixed points which we denote as I and II. First of them is trivial one, namely

Fixed Point I: $w_* = 1/u_* = g'_* = 0$

with $\gamma_1 = 0$. The matrix Ω is diagonal with the elements (eigenvalues): $\Omega_1 = \eta$, $\Omega_2 = \eta - 2\varepsilon$. Thus, the corresponding scaling regime is IR stable if $\eta > 0$ and, at the same time, $\eta > 2\varepsilon$.

The second one is defined as follows

Fixed Point II: $w_* = 1/u_* = 0$, and

$$\bar{g}'_* = \frac{2d}{d-1+\alpha}(2\varepsilon - \eta), \tag{14}$$

where we denote $\bar{g}' = g'S_d/(2\pi)^d$. The anomalous dimension γ_1 at the fixed point is $\gamma_1^* = 2\varepsilon - \eta$. These are exact one-loop results. It is the consequence of the non-existence of the higher-loop corrections. Corresponding matrix of the first derivatives is triangular with diagonal elements (eigenvalues) $\Omega_1 = 2(\eta - \varepsilon)$ and $\Omega_2 = 2\varepsilon - \eta$. Thus, the conditions $g'_* > 0$ and $\Omega_{1,2} > 0$ for the IR stable regime lead to the inequalities: $\eta > \varepsilon$ and $\eta < 2\varepsilon$.

The second nontrivial limit of our general model is so-called frozen limit given by $u \to 0$. In this case, it is again necessary to make transformation to new variables, namely, $g'' \equiv g/u$ and u is unchanged. It leads to the changes in the β functions which are now define as follows (β_u is the same as in the general case)

$$\beta_{g''} = g''(-2\varepsilon + 2\gamma_1 - 2\gamma_2), \quad \beta_u = u(-\eta + \gamma_1). \tag{15}$$

In this case we have again two fixed points which we denote as III and IV. First of them is trivial one, namely

Fixed Point III: $u_* = g''_* = 0$.
The corresponding matrix Ω is diagonal with the elements: $\Omega_1 = -2\varepsilon$ and $\Omega_2 = -\eta$. It means that the corresponding scaling regime is IR stable if $\varepsilon < 0$ and $\eta < 0$.

The second one is defined as follows

Fixed Point IV: $u_* = 0$, and

$$\bar{g}''_* = -\frac{\varepsilon}{2(A''_{10} - A''_{20})} - \frac{C''_{10} - C''_{20}}{2(A''_{10} - A''_{20})^3}\varepsilon^2, \tag{16}$$

where A''_{i0} and C''_{i0} for $i = 1, 2$ are expressions from (9) taken in the given limit and again $\bar{g}'' = g'' S_d/(2\pi)^d$. In this case we have the relation $\gamma_1^* = \gamma_2^* + \varepsilon$ between anomalous dimensions. The matrix of the first derivatives is triangular with eigenvalues (diagonal elements):

$$\Omega_1 = -\eta + \gamma_1^*, \quad \Omega_2 = 2g''_* \left(\frac{\partial \gamma_1}{\partial g''} - \frac{\partial \gamma_2}{\partial g''}\right)_* . \tag{17}$$

In the helical case, when helicity parameter $\rho > 0$, one must work with $d = 3$, and we obtain

$$\bar{g}''_* = 3\varepsilon + \varepsilon^2 \left(\frac{3}{2} - \frac{7}{4}\alpha - \frac{9}{32}\pi^2\rho^2\right), \tag{18}$$

$$\Omega_1 = \varepsilon - \eta - \frac{\varepsilon\alpha}{2} + \frac{\varepsilon^2\alpha}{192}(-64 + 56\alpha + 9\pi^2\rho^2), \quad \Omega_2 = \varepsilon\left(2 + \varepsilon\left(-1 + \frac{7}{6}\alpha + \frac{3}{16}\pi^2\rho^2\right)\right), \tag{19}$$

therefore, the conditions to have $\bar{g}''_* > 0$ and $\Omega_{1,2} > 0$ are

$$\varepsilon > 0, \quad \varepsilon\left(-1 + \frac{7}{6}\alpha + \frac{3}{16}\pi^2\rho^2\right) > -2, \quad \eta < \varepsilon - \frac{\varepsilon\alpha}{2} + \frac{\varepsilon^2\alpha}{192}(-64 + 56\alpha + 9\pi^2\rho^2). \tag{20}$$

These inequalities define the region in the parameter space for which the scaling regime is IR stable.

The last but the most interesting scaling regime is obtained when one assume that $0 < u_* < \infty$. Let us briefly discuss this case. The corresponding IR fixed point will be denoted as V. The coordinates of the fixed point is now defined by the requirement of vanishing of the β functions which are given in (12). The fixed point value for g is given as

Fixed Point V: $u_* > 0$

$$\text{FPV}: \quad g_* = -\frac{\varepsilon}{2(A_1 - A_2)} - \frac{C_1 - C_2}{2(A_1 - A_2)^3}\varepsilon^2, \tag{21}$$

where the functions A_1, A_2, C_1, and C_2 are given in (9), and where the parameter u is taken at its fixed point value u_* which is defined implicitly by the equation

$$-\eta + \gamma_1^*(u_*) = 0. \tag{22}$$

The analysis of the problem is relatively simple in one-loop level, where explicit expressions can be found [13]. On the other hand, as was briefly discussed in [18], the situation is essentially more complicated when we are working in two-loop approximation. Detail analysis shows that the investigation of the IR stability of the fixed point in the general case of the present model has to be done individually for concrete situation which is rather cumbersome and it must be done in a separate work.

Conclusions

In present paper we have studied the influence of compressibility and helicity of the system on the possible IR scaling regimes of the model of a passive scalar advected by a Gaussian velocity field with finite time correlations by means of the field theoretic RG technique. The dependence of the fixed

points, which are directly related to the existence of possible IR scaling regimes, on the parameters of compressibility and helicity as well as their IR stability is discussed. The explicit inequalities, which define the stable IR scaling regimes, are found in the case of the frozen limit of the model. The most general case with finite time correlations of the velocity field is more complicated within two-loop approximation and has to be consider in detail once more.

ACKNOWLEDGEMENTS — It is a pleasure to thank the Organizing Committee of the STM-2005 for kind hospitality. The work was supported in part by VEGA grant 6193 of Slovak Academy of Sciences, and by Science and Technology Assistance Agency under contract No. APVT-51-027904.

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